Supplementary Appendix

Intraday Stochastic Volatility in Discrete Price Changes: the Dynamic Skellam Model *

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Abstract

We present the Supplementary Appendix for our paper "Intraday Stochastic Volatility in Discrete Price Changes: the Dynamic Skellam Model" by Koopman, Lit & Lucas (2016). This Appendix contains (D) a generalisation of the dynamic modified Skellam model, (E) the details of the NAIS method for parameter estimation and for signal extraction, and (F) a range of additional Figures and Tables with empirical results, robustness checks, and model extensions. We have adopted the same notation and use of acronyms as introduced in the main paper.

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D Skellam model with time-varying mean and variance

We model the dynamics of μ_t and σ_t^2 by a (possibly) nonlinear transformation of an autoregressive process,

$$\begin{pmatrix} \mu_t \\ \sigma_t^2 \end{pmatrix} = s(\boldsymbol{\theta}_t), \qquad \boldsymbol{\theta}_t = \boldsymbol{c}_t + \boldsymbol{Z}_t \boldsymbol{\alpha}_t, \tag{D.1}$$

$$\boldsymbol{\alpha}_{t+1} = \boldsymbol{d}_t + \boldsymbol{T}_t \boldsymbol{\alpha}_t + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \sim \text{NID}(\boldsymbol{0}, \boldsymbol{Q}_t), \tag{D.2}$$

for $t = 1, \ldots, n$, where vector $s(\cdot)$ is referred to as the link function, $\boldsymbol{\theta}_t \in \mathbb{R}^{r \times 1}$ is the signal vector, with $r = 2$, $\boldsymbol{\alpha}_t \in \mathbb{R}^{m \times 1}$ is the state vector, $\boldsymbol{c}_t \in \mathbb{R}^{r \times 1}$ is a scalar intercept, $\boldsymbol{d}_t \in \mathbb{R}^{m \times 1}$ is a vector of intercepts, $\boldsymbol{Z}_t \in \mathbb{R}^{r \times m}$ is a matrix of coefficients, $\boldsymbol{T}_t \in \mathbb{R}^{m \times m}$ is a transition matrix, and the disturbances $\boldsymbol{\eta}_t$ are normally and independently distributed (NID) with mean zero and variance matrix $\boldsymbol{Q}_t \in \mathbb{R}^{m \times m}$. The vectors $\boldsymbol{c}_t$, $\boldsymbol{d}_t$ and matrices $\boldsymbol{Z}_t$, $\boldsymbol{T}_t$, $\boldsymbol{Q}_t$ are typically constant but possibly time-varying in a deterministic manner. Typical examples of link functions $s(\cdot)$ are the exponential function (to ensure positivity) and the scaled logistic function (to preserve lower and upper bounds). When the link function $s(\cdot)$ directly requires the state vector $\boldsymbol{\alpha}_t$ as an argument, we simply set $r = m$, $\boldsymbol{c}_t = \boldsymbol{0}$, and $\boldsymbol{Z}_t = \mathrm{I}_{\boldsymbol{m}}$. For an application with an observation distribution that only requires a time-varying mean or variance, we have a univariate signal and $r = 1$. The initial conditions for the elements of the state vector $\boldsymbol{\alpha}_1$ depend on their dynamic properties. The variance matrix $\boldsymbol{Q}_t$ is possibly positive semi-definite.

The model specified in equations (D.1)–(D.2) allows for a wide variety of dynamic patterns in μ_t and σ_t^2, including autoregressive moving average dynamics, time-varying seasonal and cyclical patterns, deterministic and stochastic trends, and their combinations. Regression and intervention effects can be added to the signal as well. More details of their formulations in the form of (D.2) are provided in Durbin and Koopman (2012, Ch. 3). The dynamic Skellam model as specified above falls within the class of non-Gaussian nonlinear state space models which can be represented as

$$y_t \sim p(y_t|\boldsymbol{\theta}_t; \boldsymbol{\psi}), \qquad \boldsymbol{\theta}_t = \boldsymbol{c}_t + \boldsymbol{Z}_t \boldsymbol{\alpha}_t, \qquad \boldsymbol{\alpha}_{t+1} \sim p_g(\boldsymbol{\alpha}_{t+1}|\boldsymbol{\alpha}_t; \boldsymbol{\psi}), \qquad t = 1, \ldots, n, \tag{D.3}$$

with $\boldsymbol{\alpha}_1 \sim p_g(\boldsymbol{\alpha}_1; \boldsymbol{\psi})$, where $\boldsymbol{\psi}$ is an unknown and fixed parameter vector gathering all the parameters in $\boldsymbol{c}_t$, $\boldsymbol{Z}_t$, $\boldsymbol{d}_t$, $\boldsymbol{T}_t$, and $\boldsymbol{Q}_t$, and possibly in the signal function $s(\cdot)$. The

observation density $p(y_t|\boldsymbol{\theta}_t;\boldsymbol{\psi})$ refers to the dynamic (possibly modified) Skellam distribution with signal $\boldsymbol{\theta}_t$ representing the dynamic mean μ_t and/or variance σ_t^2, see the main paper for details. The updating Gaussian state density $p_g(\boldsymbol{\alpha}_{t+1}|\boldsymbol{\alpha}_t;\boldsymbol{\psi})$ refers to the linear Markov process (D.2), and $p_g(\boldsymbol{\alpha}_1;\boldsymbol{\psi})$ represents the initial condition for $\boldsymbol{\alpha}_1$. We assume that for given realizations of the signal $\boldsymbol{\theta}' = (\boldsymbol{\theta}_1', \ldots, \boldsymbol{\theta}_n')$ the observations $\boldsymbol{y} = (y_1, \ldots, y_n)'$ are conditionally independent, and also write $\boldsymbol{\theta} = \boldsymbol{c} + \boldsymbol{Z}\boldsymbol{\alpha}$ with $\boldsymbol{c}' = (\boldsymbol{c}_1', \ldots, \boldsymbol{c}_n')$, $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1', \ldots, \boldsymbol{\alpha}_n')'$, and $\boldsymbol{Z}$ a block-diagonal matrix with blocks $\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n$ on the leading diagonal. The joint conditional density for all observations and the marginal density for all states can now be written as

$$p(\boldsymbol{y}|\boldsymbol{\theta};\boldsymbol{\psi}) = \prod_{t=1}^{n} p(y_t|\boldsymbol{\theta}_t;\boldsymbol{\psi}), \qquad p_g(\boldsymbol{\alpha};\boldsymbol{\psi}) = p_g(\boldsymbol{\alpha}_1;\boldsymbol{\psi}) \prod_{t=2}^{n} p_g(\boldsymbol{\alpha}_t|\boldsymbol{\alpha}_{t-1};\boldsymbol{\psi}), \tag{D.4}$$

respectively. Given the linear dependence of $\boldsymbol{\theta}$ on $\boldsymbol{\alpha}$, the density $p_g(\boldsymbol{\theta};\boldsymbol{\psi})$ can be constructed directly from $p_g(\boldsymbol{\alpha};\boldsymbol{\psi})$.

The state space representation implied by equations (D.3) or (D.4) for the dynamic Skellam model allows us to build on a well developed framework for the parameter estimation of $\boldsymbol{\psi}$, for the signal extraction of $\boldsymbol{\theta}$ and the filtering and smoothing of $\boldsymbol{\alpha}$; see Durbin and Koopman (2012) for a textbook treatment. As for all non-Gaussian nonlinear state space models, the main complication for the dynamic Skellam model is that the likelihood function $\int p(\boldsymbol{y}|\boldsymbol{\theta};\boldsymbol{\psi})p_g(\boldsymbol{\alpha};\boldsymbol{\psi})\,\mathrm{d}\boldsymbol{\alpha}$ is analytically intractable. We therefore adopt the method of Monte Carlo maximum likelihood for parameter estimation, but also for signal extraction. In particular, we apply the numerically accelerated importance sampling (NAIS) method of Koopman, Lucas, and Scharth (2014) and show that it can efficiently handle long univariate time series (large n). If we require a time-varying μ_t or σ_t^2, that is a univariate signal, $r = 1$, we can apply the NAIS method without extensions. For handling both a time-varying mean μ_t and variance σ_t^2, we have developed a bivariate extension of the NAIS methodology available in the Supplementary Appendix E. For the dataset in this paper we find we can set $\mu_t = 0$, such that we only consider a stochastic time-varying variance σ_t^2.

E Monte Carlo likelihood and parameter estimation

We first present a short review of Monte Carlo techniques for likelihood evaluation for state space models and the efficient importance sampler (EIS) of Liesenfeld and Richard (2003) and Richard and Zhang (2007). Next, we present our bivariate extension of the numerically accelerated importance sampling (NAIS) method of Koopman et al. (2014).

NAIS : likelihood evaluation and importance sampling

We can express the likelihood function for the non-Gaussian nonlinear state space model (D.3) as

$$L(\boldsymbol{y};\boldsymbol{\psi}) = \int p(\boldsymbol{y},\boldsymbol{\theta};\boldsymbol{\psi})\,\mathrm{d}\boldsymbol{\theta} = \int p(\boldsymbol{y}|\boldsymbol{\theta};\boldsymbol{\psi})p_g(\boldsymbol{\theta};\boldsymbol{\psi})\,\mathrm{d}\boldsymbol{\theta}. \tag{E.1}$$

An analytical expression is not available for this high dimensional integral. In cases where the model is linear and Gaussian, the Kalman filter can be used for likelihood evaluation, signal extraction and forecasting. Here we rely on numerical integration techniques that need to be both practical and feasible. It is well established that we can use Monte Carlo simulation methods for the evaluation of (E.1); see Ripley (1987) for a general introduction. A naive Monte Carlo estimate of $L(\boldsymbol{y};\boldsymbol{\psi})$ is given by

$$\frac{1}{S}\sum_{k=1}^{S} p(\boldsymbol{y}|\boldsymbol{\theta}^{(k)};\boldsymbol{\psi}), \qquad \boldsymbol{\theta}^{(k)} \sim p_g(\boldsymbol{\theta};\boldsymbol{\psi}),$$

where S is the number of Monte Carlo replications and the simulated value of $\boldsymbol{\theta}^{(k)}$ is obtained by simulating the state vectors from the vector autoregressive process (D.2) and with $\boldsymbol{\theta} = \boldsymbol{c} + \boldsymbol{Z}\boldsymbol{\alpha}$ for a given parameter vector $\boldsymbol{\psi}$. This Monte Carlo estimate is numerically highly inefficient since the simulated paths have no support from $\boldsymbol{y}$.

In various contributions in statistics and econometrics it is argued that (E.1) can be evaluated efficiently using the method of importance sampling; see, for example, Shephard and Pitt (1997), Durbin and Koopman (1997), Liesenfeld and Richard (2003) and Richard and Zhang (2007). For a feasible implementation of this method we require a Gaussian importance density $g(\boldsymbol{\theta}|\boldsymbol{y};\boldsymbol{\psi}^*)$ from which the $\boldsymbol{\theta}$s are sampled conditional on the observation vector $\boldsymbol{y}$, where $\boldsymbol{\psi}^*$ denotes a fixed parameter vector, containing $\boldsymbol{\psi}$ as well as parameters $\tilde{\boldsymbol{\psi}}$ particular to the importance density $g(\boldsymbol{y}|\boldsymbol{\theta};\tilde{\boldsymbol{\psi}})$, i.e., $\boldsymbol{\psi}^* = (\boldsymbol{\psi}', \tilde{\boldsymbol{\psi}}')'$. Under the assumption that a numerically efficient device can be developed for sampling $\boldsymbol{\theta}$ from $g(\boldsymbol{\theta}|\boldsymbol{y};\boldsymbol{\psi}^*)$, we can

express the likelihood function (E.1) in terms of the importance density as

$$L(\boldsymbol{y};\boldsymbol{\psi}) = \int \frac{p(\boldsymbol{y},\boldsymbol{\theta};\boldsymbol{\psi})}{g(\boldsymbol{\theta}|\boldsymbol{y};\boldsymbol{\psi}^*)} g(\boldsymbol{\theta}|\boldsymbol{y};\boldsymbol{\psi}^*)\,\mathrm{d}\boldsymbol{\theta}, \tag{E.2}$$

with the importance sampling estimate given by

$$\frac{1}{S}\sum_{k=1}^{S} \omega(\boldsymbol{y},\boldsymbol{\theta}^{(k)};\boldsymbol{\psi}^*), \qquad \omega(\boldsymbol{y},\boldsymbol{\theta};\boldsymbol{\psi}^*) = \frac{p(\boldsymbol{y},\boldsymbol{\theta};\boldsymbol{\psi})}{g(\boldsymbol{\theta}|\boldsymbol{y};\boldsymbol{\psi}^*)}, \qquad \boldsymbol{\theta}^{(k)} \sim g(\boldsymbol{\theta}|\boldsymbol{y};\boldsymbol{\psi}^*), \tag{E.3}$$

where the number of simulations S should be sufficiently high and where $\boldsymbol{\theta}^{(k)}$ is drawn independently for $k = 1,\ldots,S$. In this framework we assume that $p_g(\boldsymbol{\theta};\boldsymbol{\psi}) = g(\boldsymbol{\theta};\boldsymbol{\psi})$, which implies that the marginal stochastic properties of $\boldsymbol{\theta}$ in the model are the same as in the importance sampling distribution. It follows immediately that

$$\omega(\boldsymbol{y},\boldsymbol{\theta};\boldsymbol{\psi}^*) = \frac{p(\boldsymbol{y},\boldsymbol{\theta};\boldsymbol{\psi})}{g(\boldsymbol{\theta}|\boldsymbol{y};\boldsymbol{\psi}^*)} = \frac{p(\boldsymbol{y}|\boldsymbol{\theta};\boldsymbol{\psi})p_g(\boldsymbol{\theta};\boldsymbol{\psi})}{g(\boldsymbol{y}|\boldsymbol{\theta};\tilde{\boldsymbol{\psi}})g(\boldsymbol{\theta};\boldsymbol{\psi})/g(\boldsymbol{y};\boldsymbol{\psi}^*)} = g(\boldsymbol{y};\boldsymbol{\psi}^*)\frac{p(\boldsymbol{y}|\boldsymbol{\theta};\boldsymbol{\psi})}{g(\boldsymbol{y}|\boldsymbol{\theta};\tilde{\boldsymbol{\psi}})}, \tag{E.4}$$

see, for example, Durbin and Koopman (2012). The density $g(\boldsymbol{y};\boldsymbol{\psi}^*)$ can be taken as a scaling function since it does not depend on $\boldsymbol{\theta}$. The function $\omega(\boldsymbol{y},\boldsymbol{\theta};\boldsymbol{\psi}^*)$ is usually referred to as the importance sampling weight function. If the variance of $\omega(\boldsymbol{y},\boldsymbol{\theta};\boldsymbol{\psi}^*)$ exists, the estimate (E.3) is consistent for any $g(\boldsymbol{y}|\boldsymbol{\theta};\tilde{\boldsymbol{\psi}})$ and a central limit theorem applies; see Geweke (1989) and Koopman, Shephard, and Creal (2009). We may expect that a well-behaved weight function leads to an efficient importance sampling estimate of the likelihood function.

NAIS : construction of the importance density

The key choice in selecting an importance density $g(\boldsymbol{\theta}|\boldsymbol{y};\boldsymbol{\psi}^*)$ is numerical efficiency. We follow the predominant choice in the literature and opt for the Gaussian density; we construct $g(\,\cdot\,)$ efficiently using standard techniques such as regression analysis and the Kalman filter.

Several proposals for constructing a Gaussian $g(\boldsymbol{\theta}|\boldsymbol{y};\boldsymbol{\psi}^*)$ have been developed. Shephard and Pitt (1997) and Durbin and Koopman (1997) determine the choice of $\tilde{\boldsymbol{\psi}}$ via a second order Taylor expansion of density $p(\boldsymbol{y}|\boldsymbol{\theta};\boldsymbol{\psi})$ around a $\boldsymbol{\theta}$ that is equal to the mode of $p(\boldsymbol{\theta}|\boldsymbol{y};\boldsymbol{\psi})$. The mode can be found by an iterative method involving the Kalman filter and the related smoother. Alternatively, in the EIS method of Liesenfeld and Richard (2003) and Richard

and Zhang (2007), the appropriate Gaussian importance density is found by solving

$$\underset{\tilde{\boldsymbol{\psi}}_t}{\operatorname{argmin}} \int \lambda^2(y_t, \boldsymbol{\theta}_t; \boldsymbol{\psi}^*)\, \omega_t(y_t, \boldsymbol{\theta}_t; \boldsymbol{\psi}^*)\, g(\boldsymbol{\theta}_t|\boldsymbol{y}; \boldsymbol{\psi}^*)\, \mathrm{d}\boldsymbol{\theta}_t, \tag{E.5}$$

for each $t = 1, \ldots, n$, with $\tilde{\boldsymbol{\psi}}' = (\tilde{\boldsymbol{\psi}}_1', \ldots, \tilde{\boldsymbol{\psi}}_n')$, $\boldsymbol{\psi}^{*\prime} = (\boldsymbol{\psi}', \tilde{\boldsymbol{\psi}}')$, and

$$\lambda(y_t, \boldsymbol{\theta}_t; \boldsymbol{\psi}^*) := \log\, \omega_t(y_t, \boldsymbol{\theta}_t; \boldsymbol{\psi}^*) := \log\, p(y_t|\boldsymbol{\theta}_t; \boldsymbol{\psi}) - \log\, g(y_t|\boldsymbol{\theta}_t; \tilde{\boldsymbol{\psi}}_t). \tag{E.6}$$

The importance density is effectively determined by the minimization of the variance of the log weight ω_t, for each t. Richard and Zhang (2007) evaluate the integral in (E.5) using importance sampling and perform its minimization via weighted least squares regression. Koopman, Lit, and Nguyen (2012) show that the EIS method can also fully rely on computationally efficient Kalman filter and smoothing methods. Their modification leads to a faster and efficient importance sampling method, especially for large state dimensions.

In a further development of EIS, Koopman et al. (2014) replace the evaluation of the integral in (E.5) by standard Gauss-Hermite quadrature methods. This results in a highly numerically efficient importance sampling technique, that can be augmented with easy-to-compute control variates to increase efficiency even further. They label their method numerically accelerated importance sampling (NAIS). The key to NAIS is the availability of analytic expressions for the marginal densities $g(\boldsymbol{\theta}_t|\boldsymbol{y}; \boldsymbol{\psi}^*)$ given the Gaussian importance densities $g(\boldsymbol{y}|\boldsymbol{\theta}; \tilde{\boldsymbol{\psi}})$ and a Gaussian marginal density $g(\boldsymbol{\theta}; \boldsymbol{\psi}) = p_g(\boldsymbol{\theta}; \boldsymbol{\psi})$. Although NAIS was originally developed for a univariate signal $\theta_t \in \mathbb{R}$, the method can easily be extended to multiple dimensions; see Scharth (2012, Ch. 5) and the discussions in Koopman et al. (2014). Scharth (2012) proposes Halton sequences and quasi-Monte Carlo integration for the evaluation of high dimensional integrals. In the case of our dynamic Skellam model, the signal is only two-dimensional and hence we can still rely on Gauss-Hermite quadrature methods efficiently.

NAIS : bivariate numerically accelerated importance sampling

To facilitate the exposition, we express the Gaussian density as a kernel function in $\boldsymbol{\theta}_t$,

$$g(\boldsymbol{y}|\boldsymbol{\theta}; \tilde{\boldsymbol{\psi}}) = \prod_{t=1}^{n} g(y_t|\boldsymbol{\theta}_t; \tilde{\boldsymbol{\psi}}_t), \qquad g(y_t|\boldsymbol{\theta}_t; \tilde{\boldsymbol{\psi}}_t) = \exp\left(a_t + \boldsymbol{b}_t'\boldsymbol{\theta}_t - \frac{1}{2}\boldsymbol{\theta}_t'\boldsymbol{C}_t\boldsymbol{\theta}_t\right), \tag{E.7}$$

with scalar a_t, 2×1 vector $\boldsymbol{b}_t$, a symmetric 2×2 matrix $\boldsymbol{C}_t$, and bivariate $\boldsymbol{\theta}_t = (\theta_{1t}, \theta_{2t})'$. To ensure that $g(y_t|\boldsymbol{\theta}_t; \tilde{\boldsymbol{\psi}}_t)$ integrates to one, we set $a_t = -\log 2\pi + \frac{1}{2}\log|\boldsymbol{C}_t| - \frac{1}{2}\boldsymbol{b}_t'\boldsymbol{C}_t^{-1}\boldsymbol{b}_t$. We gather the five remaining parameters in $\boldsymbol{b}_t$ and $\boldsymbol{C}_t$ into the vector $\tilde{\boldsymbol{\psi}}_t$. NAIS obtains the importance sampling parameters $\tilde{\boldsymbol{\psi}}_t$ iteratively, starting from an initial guess $\tilde{\boldsymbol{\psi}}_t^{(0)}$, and updating it sequentially to $\tilde{\boldsymbol{\psi}}_t^{(k)}$ for $k = 1, 2, \ldots$, until convergence. Given $\tilde{\boldsymbol{\psi}}_t^{(k)}$, the next parameter vector $\tilde{\boldsymbol{\psi}}_t^{(k+1)}$ for the importance densities solves the EIS criterion

$$\underset{\tilde{\boldsymbol{\psi}}_t^{(k+1)}}{\operatorname{argmin}} \int\int \lambda^2(y_t, \boldsymbol{\theta}_t; \boldsymbol{\psi}^{*(k+1)}) \omega_t(y_t, \boldsymbol{\theta}_t; \boldsymbol{\psi}^{*(k)}) g(\boldsymbol{\theta}_t|\boldsymbol{y}; \boldsymbol{\psi}^{*(k)})\, d\theta_{1t}\, d\theta_{2t}, \tag{E.8}$$

where $\boldsymbol{\psi}^{*(k)}$ contains $\boldsymbol{\psi}$ and $\tilde{\boldsymbol{\psi}}^{(k)}$. The key to the implementation of NAIS is the availability of an analytical expression for the smoothing density $g(\boldsymbol{\theta}_t|\boldsymbol{y}; \boldsymbol{\psi}^{*(k)})$. In our case of Gaussian importance sampling distributions, we have

$$g(\boldsymbol{\theta}_t|\boldsymbol{y}; \boldsymbol{\psi}^{*(k)}) = \mathrm{N}(\widehat{\boldsymbol{\theta}}_t^{(k)}, \boldsymbol{V}_t^{(k)}) = \frac{1}{2\pi|\boldsymbol{V}_t^{(k)}|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{\theta}_t - \widehat{\boldsymbol{\theta}}_t^{(k)})'(\boldsymbol{V}_t^{(k)})^{-1}(\boldsymbol{\theta}_t - \widehat{\boldsymbol{\theta}}_t^{(k)})\right), \tag{E.9}$$

where $\widehat{\boldsymbol{\theta}}_t^{(k)}$ and $\boldsymbol{V}_t^{(k)}$ are obtained from the Kalman filter and smoother, for given $\boldsymbol{\psi}^* = \boldsymbol{\psi}^{*(k)}$, applied to the linear Gaussian model $\boldsymbol{x}_t = \boldsymbol{\theta}_t + \boldsymbol{u}_t$ with disturbance $\boldsymbol{u}_t \sim \mathrm{N}(0, \boldsymbol{C}_t^{-1})$ and pseudo-observation $\boldsymbol{x}_t = \boldsymbol{C}_t^{-1}\boldsymbol{b}_t$, for $t = 1, \ldots, n$. It is straightforward to verify that the observation density $\prod_{t=1}^n g(\boldsymbol{x}_t|\boldsymbol{\theta}_t; \tilde{\boldsymbol{\psi}}_t)$ is equivalent to $g(\boldsymbol{y}|\boldsymbol{\theta}; \tilde{\boldsymbol{\psi}})$ in (E.7).

We numerically implement the minimization in (E.8) by the Gauss-Hermite quadrature method; see, for example, Monahan (2001). For this purpose we define

$$\varphi(y_t, \boldsymbol{\theta}_t; \tilde{\boldsymbol{\psi}}_t^{(k+1)}, \boldsymbol{\psi}^{*(k)}) = \lambda^2(y_t, \boldsymbol{\theta}_t; \boldsymbol{\psi}^{*(k+1)}) \omega_t(y_t, \boldsymbol{\theta}_t; \boldsymbol{\psi}^{*(k)}), \tag{E.10}$$

and we select a set of abscissae $\{z_i\}_{i=1}^M$ with associated Gauss-Hermite weights $h(z_i)$, for $i = 1, \ldots M$. The numerical implementation of the minimization (E.8) becomes

$$\underset{\tilde{\boldsymbol{\psi}}_t^{(k+1)}}{\operatorname{argmin}} \sum_{i=1}^M \sum_{j=1}^M w_{ij} \cdot \varphi(y_t, \tilde{\boldsymbol{z}}_{ij,t}^{(k)}; \tilde{\boldsymbol{\psi}}_t^{(k+1)}, \boldsymbol{\psi}^{*(k)}), \tag{E.11}$$

with weight $w_{ij} = h(z_i)h(z_j)\exp(\frac{1}{2}z_i^2)\exp(\frac{1}{2}z_j^2)$ and $\tilde{\boldsymbol{z}}_{ij,t}^{(k)} = \widehat{\boldsymbol{\theta}}_t + \boldsymbol{F}_t^{(k)}\boldsymbol{z}_{ij}$, where the 2×2 square root matrix $\boldsymbol{F}_t^{(k)}$ is the result of the decomposition $\boldsymbol{V}_t^{(k)} = \boldsymbol{F}_t^{(k)}\boldsymbol{F}_t^{(k)'}$ and $\boldsymbol{z}_{ij} = (z_i\,,\, z_j)'$ for $i, j = 1, \ldots, M$. In this implementation we have used the fact that $g(\tilde{\boldsymbol{z}}_{ij,t}^{(k)}|\boldsymbol{y}; \boldsymbol{\psi}^{*(k)}) \propto \exp(-\frac{1}{2}\boldsymbol{z}_{ij}'\boldsymbol{z}_{ij})$; see Koopman et al. (2014) and Scharth (2012, Ch. 5). The decomposition

of $\boldsymbol{V}_t^{(k)}$ is needed because the joint set of M^2 abscissae $\boldsymbol{z}_{ij}$, for $i, j = 1, \ldots, M$, is associated with the bivariate standard normal distribution.

We can express the minimization problem (E.11) as a standard weighted least squares computation applied to M^2 observations for the regression equation

$$\log p(y_t|\tilde{\boldsymbol{z}}_{ij,t}^{(k)}) \quad = \quad \text{constant} + \boldsymbol{\kappa}'\tilde{\boldsymbol{z}}_{ij,t}^{(k)} - \frac{1}{2}\boldsymbol{\xi}'\text{vech}(\tilde{\boldsymbol{z}}_{ij,t}^{(k)}\tilde{\boldsymbol{z}}_{ij,t}^{(k)\prime}) + \text{error}, \tag{E.12}$$

where $\boldsymbol{\kappa}$ and $\boldsymbol{\xi}$ are regression coefficient vectors and the regression weights are given by $w_{ij} \cdot \omega_t(y_t, \tilde{\boldsymbol{z}}_{ij,t}^{(k)}; \boldsymbol{\psi}^{*(k)}) \cdot g(\tilde{\boldsymbol{z}}_{ij,t}^{(k)}|\boldsymbol{y}; \boldsymbol{\psi}^{*(k)})$, and where vech($\cdot$) stacks elements of the upper triangular part of a symmetric matrix into a vector. The resulting weighted least squares estimates for $\boldsymbol{\kappa}$ and $\boldsymbol{\xi}$ yield the new values for $\boldsymbol{b}_t^{(k+1)}$ and vech$(\boldsymbol{C}_t^{(k+1)})$, respectively. Hence, new values for $\tilde{\boldsymbol{\psi}}_t^{(k+1)}$ are obtained for each $t = 1, \ldots, n$. Using these new estimates, we can determine a new $g(\boldsymbol{\theta}_t|\boldsymbol{y}; \boldsymbol{\psi}^{*(k+1)})$ in (E.9) by constructing a new time series $\boldsymbol{x}_t$ and applying the Kalman filter and smoother to the linear Gaussian model given below (E.9). In this last step we obtain new values for $\widehat{\boldsymbol{\theta}}_t^{(k+1)}$ and $\boldsymbol{V}_t^{(k+1)}$, which we require in (E.9).

This procedure is iterated until convergence. Typically, we only need a small (< 10) number of iterations for the applications in this paper. We emphasize that the regression computations can be carried out in parallel over t, leading to a very efficient implementation.

NAIS : the algorithm

The minimum of (E.11) is obtained when $\log\, p(y_t|\boldsymbol{\theta}_t; \boldsymbol{\psi}) = \log\, g(y_t|\boldsymbol{\theta}_t; \tilde{\boldsymbol{\psi}}_t)$. Therefore we regress the log Gaussian density $\log\, g(y_t|\boldsymbol{\theta}_t; \tilde{\boldsymbol{\psi}}_t)$ as given by (E.7) on the log observation density $\log\, p(y_t|\boldsymbol{\theta}_t; \boldsymbol{\psi})$ by use of weighted least squares. The regression coefficient vector at time t, Ψ_t, consists of the intercept a_t, the individual components of the 2×1 vector $\boldsymbol{b}_t$ and the 2×2 matrix $\boldsymbol{C}_t$ at time t, i.e. $\Psi_t = (a_t, \boldsymbol{\kappa}', \boldsymbol{\xi}')'$. The optimum values $\widehat{\Psi}_t$ are obtained by applying the following iterative algorithm

(i) Find appropriate starting values for $\boldsymbol{\kappa}$ and $\boldsymbol{\xi}$ with $t = 1, ..., n$ and set $s = 1$ and $\Psi_t^{(s)} = (a_t, \boldsymbol{\kappa}', \boldsymbol{\xi}')'$. In most cases the algorithm is not very sensitive to starting values so $\boldsymbol{\kappa}$ consisting of ones and $\boldsymbol{C}_t(\boldsymbol{\xi})$ set to $\boldsymbol{I}_2$ suffices.

(ii) Construct the linear Gaussian state space model with observation equation $\boldsymbol{x}_t = \boldsymbol{\theta}_t + \boldsymbol{u}_t$ with disturbance $\boldsymbol{u}_t \sim \text{N}(0, \boldsymbol{C}_t^{-1})$ and pseudo-observation $\boldsymbol{x}_t = \boldsymbol{C}_t^{-1}\boldsymbol{b}_t$, for $t = 1, \ldots, n$

and apply the Kalman filter and smoother to obtain $\widehat{\boldsymbol{\theta}}_t^{(k)}$ and $\boldsymbol{V}_t^{(k)}$ and use these to calculate $\tilde{\boldsymbol{z}}_{ij,t}^{(k)}$ as described below equation (E.11).

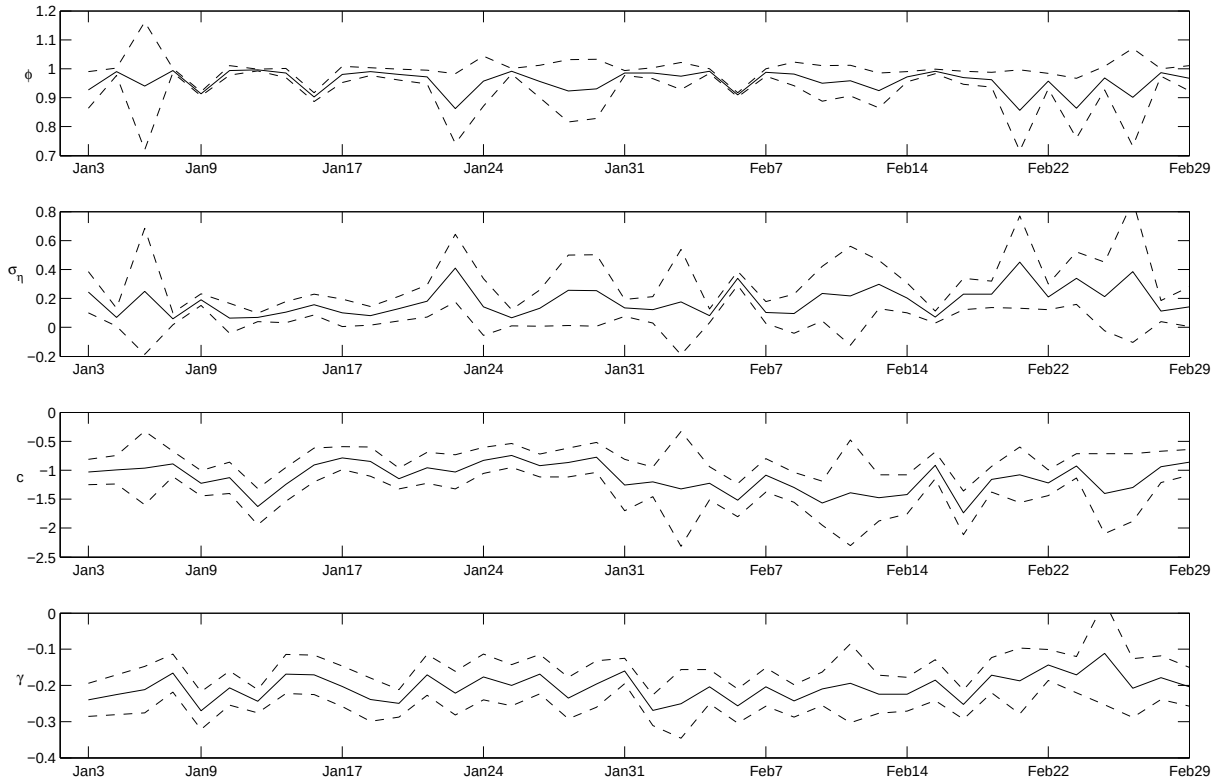
(iii) Minimize equation (E.11) by weighted least squares and obtain $\Psi_t^{(s+1)}$.

(iv) If $\sum_{t=1}^{n} ||\Psi_t^{(s+1)} - \Psi_t^{(s)}|| < \epsilon$, for some threshold value ϵ, the algorithm has converged and can be terminated. Otherwise, set $s = s + 1$ and go to step (ii).

Once the iterative algorithm has converged in step (iv), $\Psi_t^{(s+1)}, t = 1, \ldots, n$ represents the new importance density. The number of times the algorithm needs to be called before convergence depends on the model and the size of the dataset. Starting from init values the algorithm converges most of the time in 10 steps or less. For repeated analysis, values of $\widehat{\Psi}_t$ can be stored and used as starting values for the next call to the algorithm. The minimization of (E.11) can be carried out independently for all time points t and can therefore be done in parallel over t.

F Additional Figures and Tables

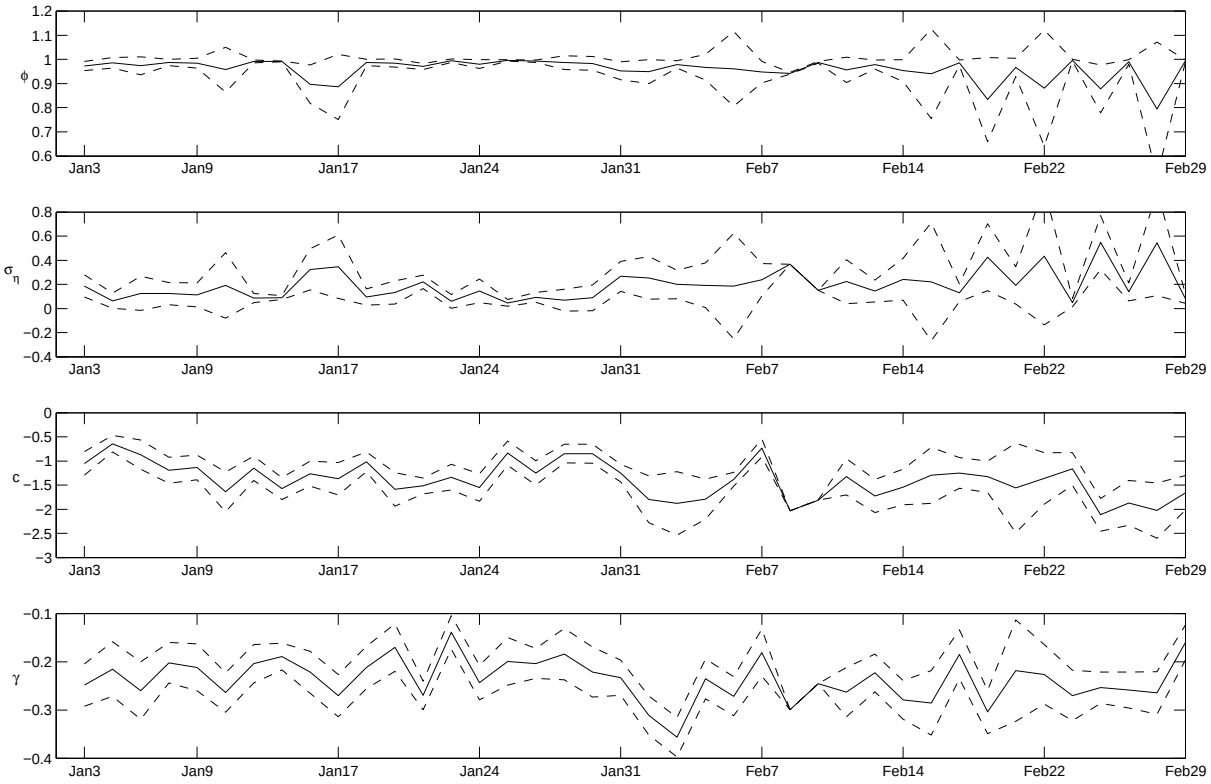
Figure F.1
Estimates of ψ with confidence intervals, WMT



The figure presents the maximum likelihood estimates of the four elements of ψ with corresponding confidence intervals based on one (asymptotic) standard error for all trading days in January 2012 and February 2012 for WMT.

Figure F.2

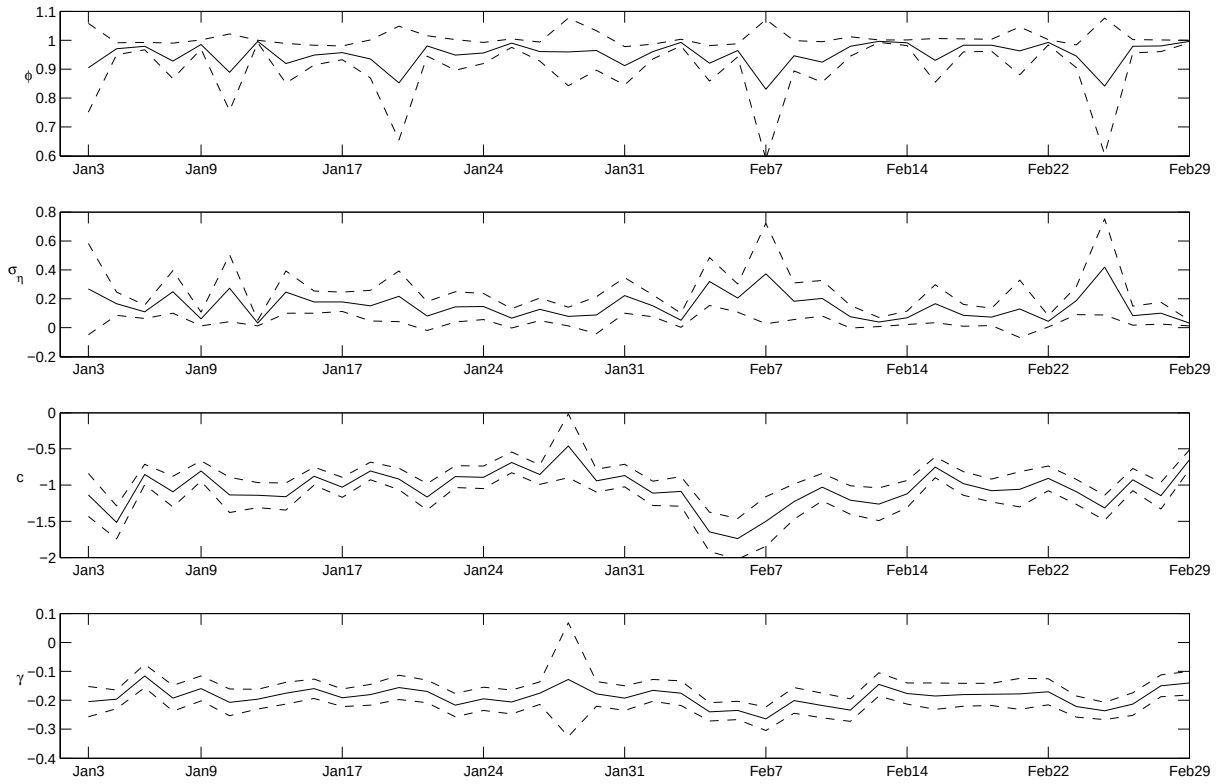
Estimates of ψ with confidence intervals, KO



The figure presents the maximum likelihood estimates of the four elements of $\boldsymbol{\psi}$ with corresponding confidence intervals based on one (asymptotic) standard error for all trading days in January 2012 and February 2012 for KO.

Figure F.3

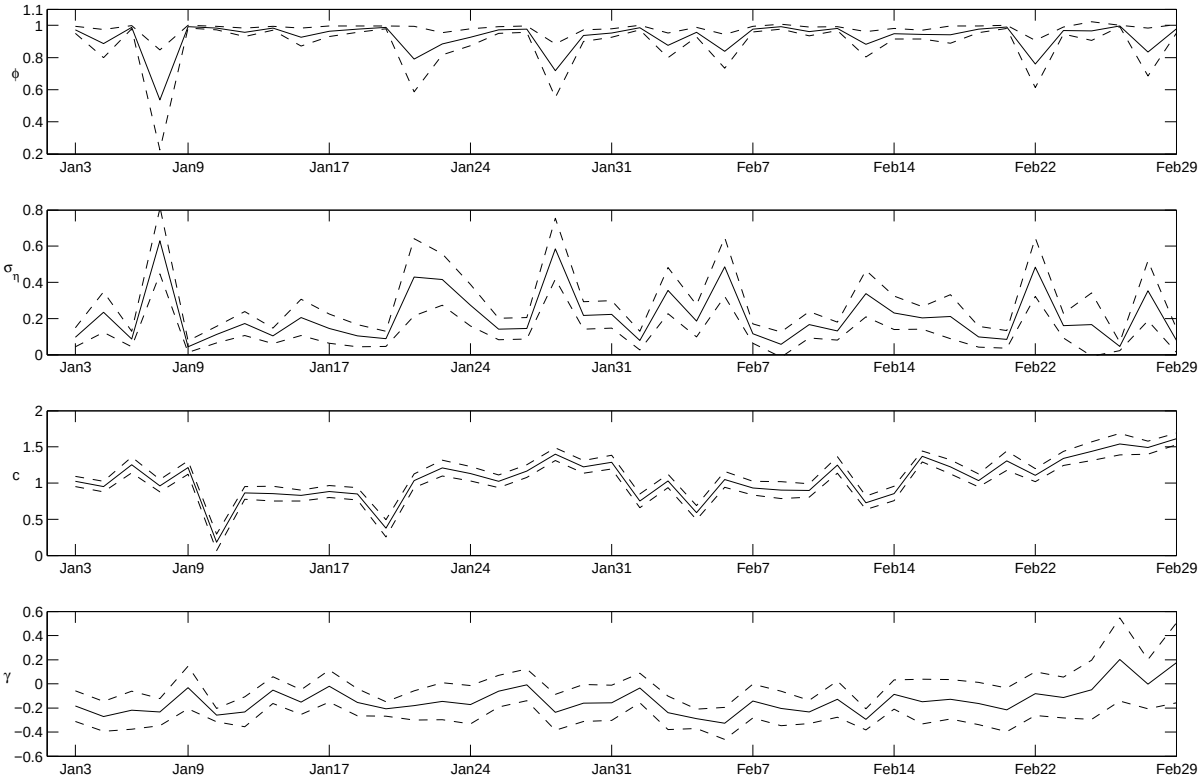
Estimates of ψ with confidence intervals, JPM



The figure presents the maximum likelihood estimates of the four elements of ψ with corresponding confidence intervals based on one (asymptotic) standard error for all trading days in January 2012 and February 2012 for JPM.

Figure F.4

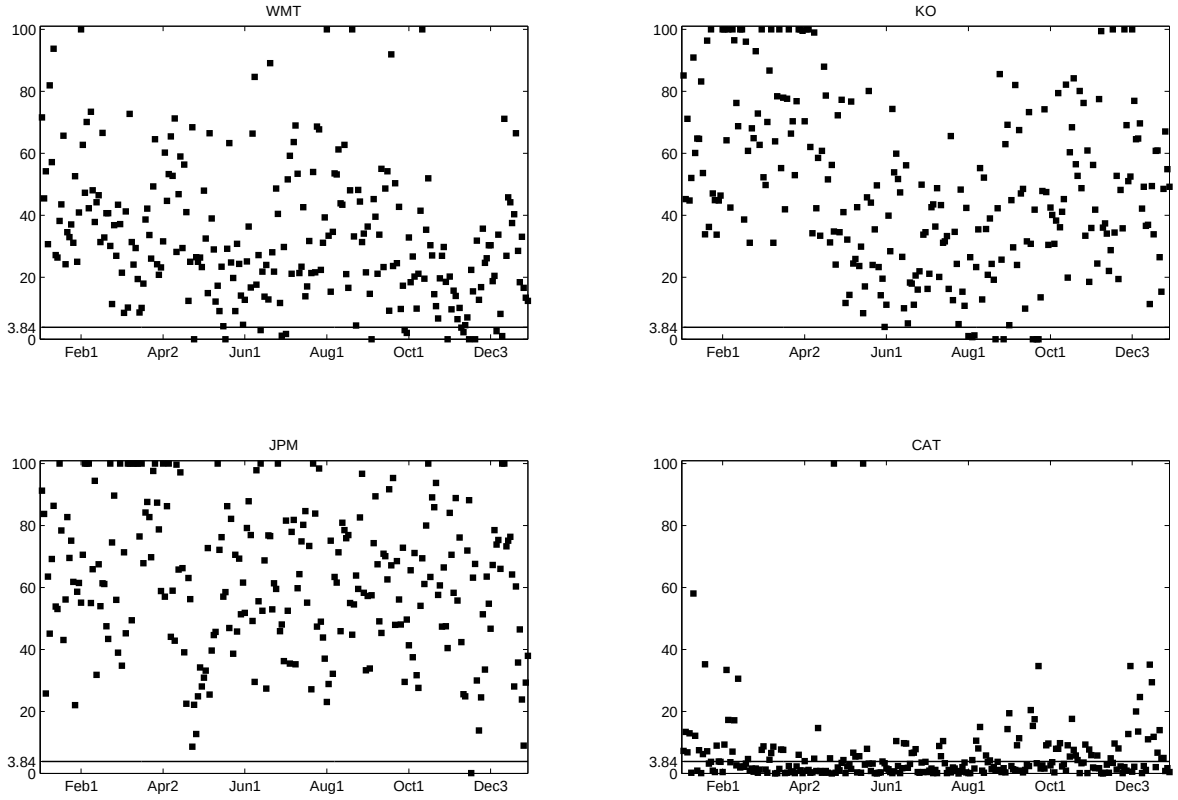
Estimates of ψ with confidence intervals, CAT



The figure presents the maximum likelihood estimates of the four elements of ψ with corresponding confidence intervals based on one (asymptotic) standard error for all trading days in January 2012 and February 2012 for CAT.

Figure F.5

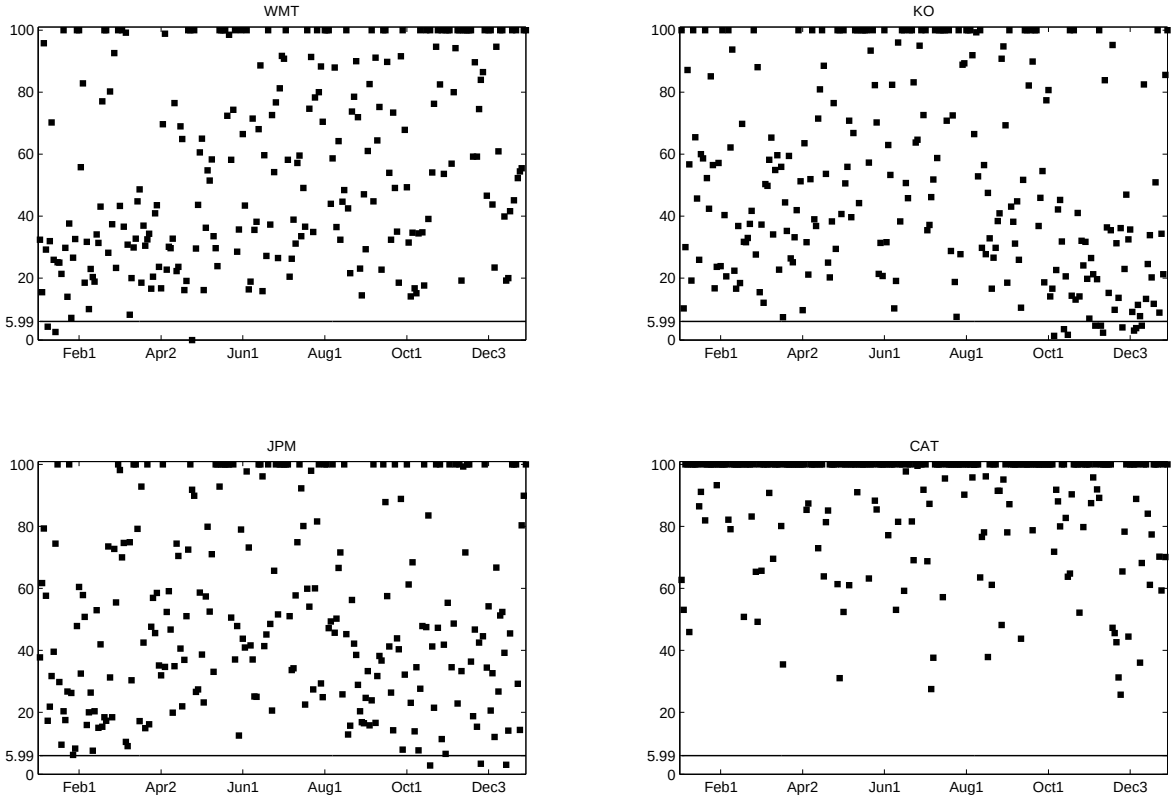
Log-likelihood ratios MSKII$(-1, 1, 0; \mu, \sigma^2, \gamma)$ and Skellam model



Each panel is for a stock and presents the log-likelihood differences (times 2) for all days in 2012. A dot indicates the log-likelihood ratio values for a specific day in 2012 between the MSKII$(-1, 1, 0; 0, \sigma^2, \gamma)$ model and the Skellam model (without the transfer of probability mass). The horizontal line indicates the 5% critical value for the $\chi^2(1)$ distribution corresponding to hypothesis H_0: $\gamma = 0$. The differences are capped at 100 for visualization purposes.

Figure F.6

Log-likelihood ratios for Model $\mathcal{B}$ (constant + spline) and Model $\mathcal{C}$ (constant + spline + autoregressive)



Each panel is for a stock and presents the log-likelihood differences (times 2) for all days in 2012. A dot indicates the log-likelihood ratio values for a specific day in 2012 between a model with only a constant and a spline $c + s_t$, Model $\mathcal{B}$, and a model with spline and autoregressive component $c + s_t + \alpha_t$, Model $\mathcal{C}$. The horizontal line indicates the 5% critical value for the $\chi^2(2)$ distribution corresponding to hypothesis H_0: $\phi = 0, \sigma_\eta = 0$. The differences are capped at 100 for visualization purposes.

Table F.1

Empirical distribution of price changes in multiples of tick size

The table reports the empirical distribution (in percentage points) of tick price changes for the four stocks Walmart, Coca-Cola, JPMorgan, and Caterpillar, in 2012. The majority of the observations are -1, 1 and 0, the distribution is close to symmetric and it centers around zero which validates the use of the MSKII$(-1, 1, 0; \mu, \sigma^2, \gamma)$ distribution.

Company	≤ -4	-3	-2	-1	0	1	2	3	≥ 4
Wal-Mart Stores Inc.	0.46	0.83	3.43	19.66	51.25	19.51	3.52	0.86	0.48
The Coca-Cola Company	0.25	0.44	2.09	18.11	58.31	17.90	2.20	0.45	0.25
JPMorgan Chase & Co.	0.15	0.40	2.42	19.37	55.29	19.29	2.53	0.41	0.14
Caterpillar Inc.	4.66	4.39	9.22	18.20	27.13	18.12	9.20	4.46	4.62

Table F.2

Expected frequencies based on the Skellam distribution

The table reports the expected frequencies (in percentage points) of tick price changes for the four stocks Walmart, Coca-Cola, JPMorgan, and Caterpillar, in 2012 based on the Skellam distribution without zero inflation/deflation. The mean absolute error (MAE) is calculated as the absolute difference between the observed and expected frequencies.

Company	≤ -4	-3	-2	-1	0	1	2	3	≥ 4	MAE
Wal-Mart Stores Inc.	0.38	1.01	4.45	18.39	51.54	18.39	4.45	1.01	0.38	0.57
The Coca-Cola Company	0.19	0.57	3.10	16.41	59.45	16.41	3.10	0.57	0.19	0.73
JPMorgan Chase & Co.	0.11	0.53	3.35	17.61	56.79	17.61	3.35	0.53	0.11	0.78
Caterpillar Inc.	4.82	5.02	9.99	17.85	24.64	17.85	9.99	5.02	4.82	0.69

Table F.3

Expected frequencies MSKII$(-1, 1, 0; \mu, \sigma^2, \gamma)$ distribution

The table reports the expected frequencies (in percentage points) of tick price changes for the four stocks Walmart, Coca-Cola, JPMorgan, and Caterpillar, in 2012 based on the MSKII$(-1, 1, 0; \mu, \sigma^2, \gamma)$ distribution. The mean absolute error (MAE) is calculated as the absolute difference between the observed and expected frequencies.

Company	≤ -4	-3	-2	-1	0	1	2	3	≥ 4	MAE
Wal-Mart Stores Inc.	0.39	0.90	3.74	20.25	49.43	20.25	3.74	0.90	0.39	0.44
The Coca-Cola Company	0.20	0.48	2.29	18.44	57.19	18.44	2.29	0.48	0.20	0.27
JPMorgan Chase & Co.	0.11	0.42	2.54	19.79	54.30	19.79	2.54	0.42	0.11	0.24
Caterpillar Inc.	4.82	4.98	9.90	18.19	24.20	18.19	9.90	4.98	4.82	0.65

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